Optimized Waveforms for Feedback Control of Vortex Shedding

Won Tae Joe, Tim Colonius, and Douglas G. MacMynowski

Abstract. Optimal control theory is combined with the numerical simulation of an incompressible viscous flow to control vortex shedding in order to maximize lift. A two-dimensional flat plate model is considered at a high angle of attack and a Reynolds number of 300. Actuation is provided by unsteady mass injection near the trailing edge and is modeled by a compact body force. The adjoint of the linearized perturbed equations is solved backwards in time to obtain the gradient of the lift to changes in actuation (the jet velocity), and this information is used to iteratively improve the controls. The optimized control waveform is nearly periodic and locked to vortex shedding. We compare the results with sinusoidal open- and closed-loop control and observe that the optimized control is able to achieve higher lift than the sinusoidal forcing with more than 50% lower momentum coefficients. The optimized waveform is also implemented in a simple closed-loop controller where the control signal is shifted or deformed periodically to adjust to the instantaneous frequency of the lift fluctuations. The feedback utilizes a narrowband filter and an Extended Kalman Filter to robustly estimate the phase of vortex shedding and achieve phase-locked, high lift flow states.

1 Introduction

Previous work on flow control over an airfoil has used periodic excitation, such as unsteady mass injection or synthetic jets, to show that the oscillatory addition of momentum can delay boundary layer separation and reattach the separated flow [7, 8], or delay dynamic stall on a rapidly pitching airfoil [10]. Unsteady actuation was also shown to change the global dynamics of vortex shedding of post-stall flow, leading to higher unsteady lift than the natural shedding [13, 17].

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In this paper, we investigate a simple model of a purely translating flat plate at high angle of attack at a Reynolds number of 300, where strong, periodic vortex shedding occurs. A small amplitude body force intended to mimic oscillatory mass injection is applied near the trailing edge in order to modulate the vortex shedding. We first consider open-loop control utilizing sinusoidal waveforms. It is observed that open-loop forcing can significantly amplify the lift, but feedback is required to tune the phase of actuation to a particular phase of the measured lift in order to lock the forcing with the phase shift associated with the highest period-averaged lift.

Rather than optimizing the phase of the control relative to the lift using only sinusoidal waveform, we investigate the possibility of optimizing the lift using more general (non-sinusoidal) actuation waveforms. We utilize a gradient-based approach that has been used previously in simulations to reduce the turbulent kinetic energy and drag of a turbulent flow in a plane channel [4], or to reduce free-shear flow noise[16]. Given the DNS for a particular actuator waveform, we solve the adjoint of the perturbed linearized equations backward in time to determine the sensitivity of the lift to the actuator input, and subsequently use this information to iteratively improve control.

This computed optimal control requires knowledge of the full flow state and therefore is not practical for real-time control. Instead, we use a period of the optimal waveform together with the previously developed phase-locking feedback strategy in order to provide a robust and practical approach to giving near-optimal performance.

In the next section, we present the simulation methodology and the actuation scheme. Results from sinusoidal forcing will be briefly summarized in Sect. 3. Once the objective of our control is defined, we formulate an adjoint-based optimization in Sect. 4. In Sect. 5, we design a feedback algorithm where the optimized waveform is shifted or deformed periodically to adjust to the output frequency of the flow. We show that the feedback controller achieves as high lift as the optimization, and can be started from any phase of the natural shedding cycle. Then the feedback control with optimized waveform is directly compared to the sinusoidal forcing case in Sect. 6. Finally, we investigate the sensitivity of the lift to the phase shift and other features of the optimized waveform.

2 Governing Equations and Numerical Method

Simulations of flow over a two-dimensional flat plate at $Re = 300$ and an angle of attack of $40^\circ$ are performed with the immersed boundary projection method combined with a vorticity-streamfunction multi-domain technique.[15, 6] We model the actuation as a point body force smeared over a few grid points with its strength defined by its velocity as $U_{jet}$. This method solves the incompressible viscous flow equations, presented here in operator form by (1). The control is implemented as a velocity boundary conditions $\phi(x,t)$ applied at the actuation points $\mathcal{E}$ shown in figure 1. In this paper, control is a function only of time, and $\phi(x,t) = \phi(t) = U_{jet}(t)$, which is the prescribed velocity at the actuation point. The operator form of our flow equations is then
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\[ N(q) = F\phi(t), \]  

where \( q \) is a vector of flow variables, 

\[ q = [\gamma \quad \vec{f}]^T \quad \text{and} \quad F = [F_\gamma \quad F_\vec{f}]^T. \]

\( \gamma \) is the discrete circulation and \( \vec{f} = [\vec{f}_x \quad \vec{f}_y] \) is a vector of surface forces on the Lagrangian body points applied to satisfy the no-slip condition for a stationary body points or the prescribed velocity for the actuation points. The vector \( F \) allocates the control action and has a single non-zero entry that corresponds to the actuator location associated with the appropriate element of the surface force, \( \vec{f} \). For brevity, we do not write out the explicit form of \( N \). Reference [6] provides the detailed equations.

This method is capable of solving for incompressible flows over an arbitrary-shaped body in motion and deformation. Here we employ this method with a stationary, rigid flat plate. In what follows, all velocities and length scales are nondimensionalized by the freestream velocity and the chord, \( U_\infty \) and \( c \), respectively.

The numerical method utilizes a series of overlapping uniform Cartesian grids of differing resolution[6]. The finest grid, encompassing the body, is comprised of a rectangular domain extending to \([-1,4] \times [-1.5,1.5]\) in the streamwise (x) and vertical (y) directions with a uniform grid spacing of 0.02 units. The constant time step was 0.004. The coarsest grid extended to \([-8,32] \times [-12,12]\]. The boundary condition at the outermost grid was that the streamfunction corresponding to the difference between the full velocity and a uniform free stream was zero. Selected cases were run on finer grids and with larger extents to demonstrate convergence and independence to far-field boundary conditions.

The lift and drag coefficient on the flat plate is defined by

\[ C_L = \frac{F_y}{\frac{1}{2}\rho U_\infty^2 c} \quad \text{and} \quad C_D = \frac{F_x}{\frac{1}{2}\rho U_\infty^2 c}, \]

where \( \rho \) is the freestream density of the fluid and \( F_y \) and \( F_x \) are lift and drag on the plate, respectively, obtained by summing over surface forces in y-direction, \( \vec{f}_y \) or in x-direction, \( \vec{f}_x \). Since the force obtained is normal to the plate and \( F_x \) is only the vertical component of the normal force, the increase of the normal force increases
both the lift and drag. As the angle of attack increases, the drag component of the normal force is increased while the lift component is reduced. For high angles of attack, this might result in decrease of the lift-to-drag ratio even in the presence of lift enhancement. However, for the purpose of demonstrating the control algorithm to achieve high lift, we will pay closer attention to the lift component of the normal force, $C_L$.

In defining the momentum injection added by the forcing, the width of the actuator is estimated as the grid spacing, $\Delta x$. The momentum coefficient, defined in Eq. (4), is the ratio between the momentum injected by the forcing and that of the freestream.

$$
C_\mu = \frac{\rho U_{jet}(t)^2 \Delta x}{\frac{1}{2} \rho U_{\infty}^2 c} \quad C_\mu' = \frac{\rho \langle U_{jet}(t) \rangle^2 \Delta x}{\frac{1}{2} \rho U_{\infty}^2 c} \tag{4}
$$

The values of $C_\mu$, $C_\mu'$ are based on the average and the root mean square of control input, $U_{jet}(t)$ and $\langle U_{jet}(t) \rangle$, respectively and the width of the actuator, $\Delta x = 0.02$.

### 3 Sinusoidal Forcing

For the translating flat plate at $Re = 300$, steady attached flow is observed for $\alpha < 10^\circ$. At $\alpha = 10^\circ$, the flow is observed to be separated but remains steady. The flow undergoes a Hopf bifurcation between angles of attack of $12^\circ$ and $15^\circ$, [5] after which vortex shedding occurs with natural shedding frequency, $\omega_n$ ($\omega_n \in [3.65, 1.39]$ for $\alpha \in [15^\circ, 50^\circ]$). Using the vertical projection of the plate to the freestream, we find that $\omega_n$ can be scaled, for $\alpha \geq 30^\circ$, to a Strouhal number of $St = f_n c \sin(\alpha)/U_{\infty} \approx 0.2$, where $f_n = \omega_n/(2\pi)$. This agrees with the wake Strouhal number for vortex shedding behind two-dimensional bluff bodies[12, 3, 9]. The unsteady shedding cycle consists of vortices of opposite signs alternately shed from the leading and trailing edges, creating periodic oscillations in the lift and drag. As $\alpha$ is increased, larger vortex structures are formed, inducing a larger amplitude of oscillation in the force exerted on the plate. For $\alpha \geq 30^\circ$, the vortex structure on the suction side of the plate is observed to be created from the leading edge and can be viewed as a transient leading-edge vortex (LEV), or, equivalently, a dynamic stall vortex (DSV) that occurs during a rapid pitch up. Maximum lift is found when the LEV is brought down to the suction side of the plate as it grows in strength. The lift decreases as the new vortex structure of the opposite sign is formed at the trailing edge. This trailing-edge vortex (TEV) pushes up the LEV sitting on the suction side of the plate, and finally halts its growth causing it to pinch-off and shed into the wake.

In order to investigate the effect of unsteady blowing on these vortex shedding cycle, we first consider sinusoidal forcing. The control is applied as a nondimensional jet velocity, $\phi = \tilde{U}_{jet} + U_{jet}(t) \sin(\omega t)$, where $\tilde{U}_{jet} = \langle U_{jet} \rangle = U_{jet}'$, thus $C_\mu = C_\mu'$. Since the goal is to maximize lift from shedding of the coherent vortex structures rather than the suppression of shedding or separation, the frequency, $\omega$, is initially
chosen to be the natural shedding frequency for each angle-of-attack \( \alpha \). The flow is phase locked to the actuation for \( \alpha \leq 15^\circ \). However, for \( \alpha \geq 20^\circ \), a subharmonic resonance is excited and over each period of forcing there is a phase shift between the forcing and lift signals. Certain phase shifts result in very high period-averaged lifts and for \( \alpha > 20^\circ \), this period-averages lift is greater than the maximum lift occurring in the baseline flow. If the feedback allows us to adjust the frequency of the actuation accordingly to keep the phase shift between the forcing signal and the lift constant, we should be able to reproduce the high-lift shedding cycles over a wide range of \( \alpha \). This feedback controller will be designed in Sect. 5 and it will be shown that the feedback can achieve the desired phase-locked shedding cycle.

4 Optimization

With the sinusoidal forcing, we can only optimize the phase of the control relative to the lift. However, continuous sinusoidal forcing could be adding circulation when it is unnecessary, or undesirable. Thus we employ an adjoint-based optimization in order to find the waveform (time history of \( U_{\text{jet}} \)) that maximizes the lift for a given actuation amplitude. We compute the optimal control over a time horizon, using the receding-horizon approach[4]. The procedure is similar to previous studies [4, 16] and is only outlined briefly here.
To maximize lift, we define a cost functional to be minimized

$$\mathcal{J} = -\int_{t_0}^{t_1} \int_{\Omega} \mathcal{F}^2(\phi(t), x, t) \, dx \, dt + C_w \int_{t_0}^{t_1} \int_{\Omega} \phi^2(t) \, dx \, dt$$  \hspace{1cm} (5)

where $t_0$ and $t_1$ are the start and end times of the optimization horizon and $\Omega$ is surface of the body (see Fig. 1). $\phi$ is the control input, in this case $\phi(t) = U_{jet}(t)$. Again, $\mathcal{F}$ is $y$ component of forces on the plate calculated in the immersed boundary projection method. The first term is the total squared lift over the optimization horizon. The second term penalizes the actuator amplitude in order to keep $C_{u\phi}$ to a value commensurate with the open-loop control discussed previously. The control weight, $C_w$, is determined by trial and error and is held fixed throughout the optimization.

At each iteration of the optimization, we modify the controls according to

$$\phi^{k+1} = \phi^k + r \cdot g(\phi^k),$$  \hspace{1cm} (6)

where $g(\phi)$ is the gradient of the cost function with respect to the controls, and $r$ is the generalized distance determined iteratively (using Brent’s line minimization) to minimize the cost function. $g(\phi)$ is found by solving

$$g(\phi) = \mathcal{N}^{\ast}(q) \Phi^* + 2C_w \phi,$$  \hspace{1cm} (7)

where $\Phi^*$ are the force unknowns in the linearized adjoint equations [1]

$$\mathcal{N}^{\ast}(q) \Phi^* = \Phi^*.$$  \hspace{1cm} (8)

Here $q^*$ are the adjoint variables (discrete circulations and forces) and $\Phi^*$ is given by

$$\Phi^* = [F_\Phi^* \ F_\mathcal{F}^*]' = [0 \ 2\mathcal{F}]'.$$  \hspace{1cm} (9)

The adjoint operator requires the full flow field from the (forward) Navier-Stokes simulation (Eq. 1) at every time step. However, in order to save memory, we saved the flow solution only every few time steps and used a linear interpolation in time. Several test cases were done with a different number of time steps skipped, including a case where the solution was saved at every time step, and no significant differences were noted between them.

All optimizations used zero control ($\phi = 0$) for the first iteration ($k = 1$) on each optimization horizon. At each iteration, we required roughly ten full Navier-Stokes simulation to perform the line minimization (to find $r$).

Optimization was done over a horizon $T = [t_0, t_1]$, where the horizon, $T$, is long enough to overcome transient effects, but limited by the computational effort to perform all the required iterations and to tune the control weight. We found for this problem that after about two periods the controls converged to an approximately periodic signal with each period corresponding to a vortex shedding cycle. A horizon of 6 periods gave the results presented below, and tests showed that the results were not very sensitive as the horizon was varied from about 5 to 8 periods. Once the iteration of the optimization converges, the control near the end of each optimization
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\[ \int_{t_0}^{t_1} \phi^2(t) \, dx \, dt \]

Optimization horizon and \( \Omega \) is given by \( \phi(t) = U_{f}(t) \).

This is to be repeated in the immersed boundary framework for lift over the optimization horizon, and in order to keep \( C_{\mu} \) to a value previously. The control weight, \( \lambda_1 \), is determined throughout the optimization.

The controls are determined according to

\[ r = F \cdot \text{control} \]

Subject to the controls, and \( r \) is determined by minimizing (Brent’s line minimization) to

\[ \sum_{j=1}^{n} \left[ \phi_j - \phi(j) \right]^2 \]

The adjoint equations [1]

\[ \phi^* = \int_{t_1}^{t_2} \frac{d}{dt} \phi^* \, dt + \frac{d}{dt} \phi^* \]

where the (forward) Navier-Stokes and adjoint Navier-Stokes equations are linearly interpolated in time.

The first iteration \( k = 1 \) on each time step is set to the beginning of the periodic cycle, and no significant differences were observed.

The first iteration \( k = 1 \) on each time step is set to the beginning of the periodic cycle, and no significant differences were observed.

Fig. 3 Schematic of receding-horizon predictive control. First the optimization of controls are performed on horizon \([t_0, t_1]\). Each iteration of optimization gives the update on control. Once the convergence of the control on the optimization is achieved, the flow is ‘advanced’ some portion \( T_a \) of the period \( T \), and controls near the end of the optimization horizon are discarded and the optimization is begun anew on horizon \([t_0 + T_a, t_1 + T_a]\).

horizon (transient of adjoint simulation) is discarded and the optimization is begun anew. This process is depicted in Fig. 3.

Optimization results in a periodic control waveform after a couple of transient periods. As shown in Fig. 6, this periodic optimal waveform is not sinusoidal, but rather composed of two distinct pulses per shedding cycle. The larger, later pulse is roughly in phase with the maximum lift. This result will be further discussed in Sect. 6 after feedback is designed to achieve highest-lift, phase-locked shedding cycle with a given optimal or sinusoidal control waveform. Different values of control weight, \( C_{\mu} \), results in a periodic control waveform with similar features, but with different average control input, thus different values of \( C_{\mu} \). For example, \( C_{\mu} = 0.3 \) gives the results shown in Fig. 6 where \( C_{\mu} \) is about two times lower than that used for the sinusoidal forcing, but comparable lift is achieved. It should be noted that, although we cannot be assured that this is a global optimal, we observed similar results with different values of control weight and different initial controls (zero, constant, or sinusoid).

5 Feedback

Feedback provides a periodic control waveform after a couple of transient periods. While it is straightforward to extract a single period of the optimal waveform, the performance can be significantly degraded if this is applied to the plate in the open loop as shown in Fig. 5. Depending on the precise state of the flow upon initiation of forcing, the flow fails to lock onto the optimal waveform or locks on with a different phase than the optimal controller. Moreover, initial transients and
subharmonic resonances further degrade the performance. Thus in this section, we design a practically implementable feedback algorithm to achieve phase lock between the lift and the optimal control waveform deduced from the adjoint-based algorithm in the previous section.

For example, we may decompose the optimal control waveform as

\[
\phi_{\text{optimal}}(t) = A_0 + \sum_{k=1}^{N_k} \left[ A_k \cos(k\omega t) + B_k \sin(k\omega t) \right]
\]

\[
= A_0 + \sum_{k=1}^{N_k} \left[ A_k \cos(k\theta(t)) + B_k \sin(k\theta(t)) \right],
\]

where \(N_k\) is the number of harmonics retained and \(\omega\) is the fundamental frequency of the optimal waveform. We used \(N_k = 10\) which provided a reasonable representation (less than 5% deviation from the original optimized waveform).

In order to implement this optimal waveform with a consistent phase difference between each of the harmonics, instantaneous phase information of the lift signal is required. The frequency of the lift signal is tracked with an Extended Kalman Filter (EKF) to estimate the phase, \(\theta(t)\) for use in Eq. 10. To improve the EKF phase estimate, narrowband filter is first used on the lift cycle to obtain a more nearly sinusoidal signal as input to the EKF. The EKF frequency estimate is then used to tune the filter to avoid introducing phase lag. The overall feedback algorithm is illustrated in Fig. 4.

![Feedback Algorithm Diagram](image)

**Fig. 4** Schematic of feedback
Thus in this section, we show how to achieve phase lock between the shedding signal and the feedback control waveform as

$$\phi = \sum_{k=1}^{\infty} \left[ A_k \sin(k\omega t) + B_k \sin(k\theta(t)) \right],$$  \hspace{1cm} (10)

where $\omega$ is the fundamental frequency of the shedding signal and $\theta(t)$ is a reasonable representation of the feedback control waveform. This approach provides a consistent phase difference throughout the cycle. The formation of the lift signal is then computed with an Extended Kalman Filter (EKF) [14].

To improve the EKF phase estimate, we introduce a low-pass filter within each cycle to obtain a more nearly instantaneous frequency estimate is then used to compute the phase. The overall feedback algorithm is summarized by

The filtered lift, $y(t)$, retains the dominant frequency, initially estimated as $\omega_0$, and filters out higher harmonics. Next, $y(t)$ is modeled as a pure sinusoid

$$\hat{y}(t) = \hat{a} \sin(\hat{\theta}(t)),$$

where $\hat{\theta}$ is estimated with the EKF; values for noise processes are chosen in the EKF so that the algorithm converges in a few cycles. Our implementation of the EKF follows closely the description in Tadmor [14] and Patoor et al. [11].

When computing $y(t)$, the initial estimate for $\omega_0$ is updated with the estimate $\hat{\omega}_0$, the frequency estimated by the EKF, and we write

$$\phi_{\text{optimal}}(t) = A_0 + \sum_{k}^{N_k} \left[ A_k \cos(k(\hat{\theta}(t) - \theta_{\text{desired}}(t))) + B_k \sin(k(\hat{\theta}(t) - \theta_{\text{desired}}(t))) \right],$$

where $\theta_{\text{desired}}$ is an additional (specified) phase shift relative to the lift signal.

Also, note that this feedback controller can be simply implemented for the sinusoidal waveform by setting $N_k = 1$, $A_1 = 0$, and $A_0 = B_1 = 0.5$ for $C_{\mu} = 0.01$. 

Fig. 5 Comparison between open-loop control case (grey) and the feedback control case (black) with optimized waveform ($N_k = 10$) at $\alpha = 40^\circ$.
6 Results of Optimized Feedback Control

As shown in Fig. 5, feedback control of the optimized waveform is able to reproduce the high-lift limit cycle that the optimization achieved, but starting from an arbitrary phase of the baseline limit cycle. The feedback system converges to something very close to the previous solution after 4 to 5 periods, and is indistinguishable after about 10 cycles.

Fig. 6 compares a few periods of the optimal control signal \( U_{\text{jet}}(t) \) and the resulting lift coefficient. The results are plotted against the closed-loop controlled case with a sinusoidal waveform where this compensator phase locked the flow at a limit cycle associated with the highest average lift at a given \( C_\mu = 0.01 \).

For both sinusoidal and optimized control, the primary effect of actuation is to create extra vorticity which is fed into the TEV as the vortices are alternately being formed and shed. After the first local minimum lift, as the new LEV is being formed, both waveforms start to feed extra circulation at the trailing edge, leading to more definite pinch off of the LEV at the following global minimum lift. The magnitude of actuation increases as the growing LEV (lift is increasing) is pulled down by the growing TEV, and finally reaches its peak near the maximum lift. The corresponding TEV is strengthened and caused to shed from the trailing edge, thereby allowing the LEV to grow larger. This results in more vertically elongated TEV that induces stronger downwash near the trailing edge, causing the LEV to sit closer to the plate, leading to higher lift compared to the baseline.

The flow field for the optimized control does not look very different from the sinusoidal forcing. However, in the optimized control, the short pause between the two pulses slows down the growth of the TEV momentarily. This separates the TEV into two structures combined by a thin vortex sheet. The effect of the dip between the two pulses will be investigated further with the feedback control in section 5.

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**Fig. 6** Comparison of optimized control (solid) with closed-loop sinusoidal forcing (dashed) at \( \alpha = 40^\circ \). Maximum and minimum lift of baseline (- - -) case is shown as a reference.
The waveform is able to reproduce the LEV, but starting from an arbitrary point, it converges to something very close to the LEV and is indistinguishable after about 15 cycles.

The control signal \(U_c(t)\) and the actuator force \(F_x\) are adjusted to phase lock the flow at a given \(C\mu = 0.01\).

The primary effect of actuation is to sharpen the TEV, and the two TEVs alternately being shed and the new LEV being formed, does not cause the trailing edge, leading to more lift and a minimum lift. The magnitude of the lift (decreasing) is pulled down by the actuator, minimum lift. The corresponding TEV is then extending, thereby allowing the TEV to elongate that induces the LEV to sit closer to the plate, and the TEV must look very different from the baseline, the short pause between the TEVs is considerably. This separates the TEV from the baseline. The effect of the dip between the TEV and feedback control in section 5.

![Diagram](image1)

![Diagram](image2)

Fig. 7 Average lift of optimized control (□) and closed-loop sinusoidal forcing (○) at different values of \(C\mu\) at \(\alpha = 40^\circ\).

The most distinct feature of the optimized control compared to sinusoidal forcing is the gradual increase in \(\phi\) during the most of the cycle followed by a more rapid decrease after its peak. A gradual addition of circulation alters the formation of the TEV such that it interferes minimally with the natural formation of LEV and only acts as a downwash to push the LEV closer to the plate. Immediately after the maximum lift, the forcing is turned off sharply. This phase of the shedding cycle is where the optimized control achieves similar magnitude of lift with minimal control input compared to the sinusoidal control. Since the shedding of the LEV is probably unavoidable in two-dimensional flow (no spanwise flux of vorticity in z-direction) after the maximum lift has been achieved, letting it shed naturally may be the most energy efficient. For the periods shown in Fig. 6, optimized control resulted in an average lift and drag coefficients of \(C_{L,ave} = 2.50\) and \(C_{D} = 2.06\), corresponding to the average lift-to-drag ratio of \(C_{L}/C_{D} = 1.20\) with \(C\mu = 0.005\) and \(C\mu' = 0.010\). With sinusoidal waveform, the feedback achieved \(C_{L} = 2.25\) and \(C_{D} = 1.83\) \((C_{L}/C_{D} = 1.20)\) with \(C\mu = C\mu' = 0.010\). Compared to baseline flow \((C_{L} = 1.35, C_{D} = 1.20,\) and \(C_{L}/C_{D} = 1.104)\), optimized control resulted in more than 85% increase in average lift. Fig. 7 compares the average lift values from the optimized control to the results from the feedback controlled cases with sinusoidal waveform where the compensator phase locked the flow at a limit cycle with the highest average lift at different values of \(C\mu\). At \(C\mu\) below 0.0065, the lift performance of the sinusoidal control decreases sharply and approaches close to the average lift of the natural flow at \(C\mu = 0.005\). However, optimized control is able to produce high lift even at low \(C\mu = 0.0025\).

The feedback controller now allows us to phase-lock an essentially arbitrary waveform, and we can utilize this fact to investigate which features of the optimized waveform are critical to high lift. In Fig. 8, we demonstrate the effect of smoothing the optimal waveform by retaining fewer harmonics in the Fourier expansion. Using \(N_k = 4\), for example, smoothes out the dip between the two highest maxima,
Fig. 8 Comparison between feedback control cases with optimized waveform at $\alpha = 40^\circ$: $N_k = 10$ (dashed) and $N_k = 4$ (solid).

Fig. 9 Maximum and minimum lift (□) and average lift (⊙) of phase-locked limit cycles at different phase shift with optimized waveform ($N_k = 10$) at $\alpha = 40^\circ$. Maximum and minimum lift of baseline (- - -) case is shown as a reference.

but has little impact on the lift achieved. This indicates that, during this phase of the shedding cycle, the sensitivity of the first term (lift-maximizing term) in Eq. 5 to the change in $\phi$ is small compared to the second term (control-penalizing term). The short pause between the two pulses may be just an energy-saving feature of the optimal control.

Fig. 9 investigates the sensitivity of the lift performance of the phase-locked limit cycles to the changes in the phase shift, $\theta_{\text{desired}}$ with the optimal control ($N_k = 10$). Feedback is able to phase lock the flow at any desired phase shift after 3 ~ 5 periods over a wide range of $\theta_{\text{desired}}$. Due to pulse-like feature of the optimal waveform, the lift is quite sensitive to changes in the phase shift, with the average lift dropping below the maximum lift of the baseline with $20^\circ$ phase changes. Because the
Optimized waveform rapidly decreases immediately after its peak, forcing with the peak prior to the maximum lift (at negative phase shift) impacts the lift significantly. Also, positive phase shift penalizes the lift performance since the magnitude of $g(\phi)$ (sensitivity of the cost functional, Eq. 5 to changes in control, $\phi$) is small during the lift-decreasing phase; thus, control is not as effective. As phase shift approaches $\pm 180^\circ$ (out of phase), the forced flow results in the average lift similar to that of an unforced flow.

7 Conclusion

A gradient-based (adjoint) approach was applied in a receding-horizon setting to optimize the control waveform in order to maximize the lift on a two-dimensional airfoil at $\alpha = 40^\circ$ and $Re = 300$. The optimized control waveform is not sinusoidal, but rather is pulse-like, with each period composed of two distinct pulses (a primary, as well as a smaller earlier pulse). It is interesting to note that pulsatile waveforms, with pulse durations much shorter than the convective time scale, have also been shown to be effective in open-loop forcing in separation control[2]. The most distinct feature of the optimal control is a gradual increase in the forcing $\phi$ during most of the cycle, followed by a more rapid decrease after its peak. This minimal control effort after the maximum lift, combined with the short pause between the two pulses provides more energy-efficient control than sinusoidal forcing. As a result, the optimal control achieves comparable lift with 2 times lower $C_\mu$ value ($C_\mu = 2.50$ with $C_\mu = 0.005$) as the sinusoidal forcing case ($C_\mu = 2.25$ with $C_\mu = 0.010$).

Optimal control provides a periodic control waveform. However, if applied in open loop, the flow fails to phase lock onto the optimal waveform, degrading the lift performance. We designed a feedback algorithm to obtain phase-locked limit cycles. Using a Fourier representation of the optimized waveform, $\phi_{\text{optimal}}$, the control parameterizes the waveform in terms of its phase $\theta(t)$, allowing the feedback to march along $\phi_{\text{optimal}}$ with consistent phase difference between each of its modes. The control consists of the following steps: 1. A narrowband filter is used on the lift cycle to obtain a more nearly sinusoidal signal. 2. The filtered lift signal is used as input to frequency tracking Extended Kalman Filter (EKF) to estimate the phase, $\hat{\theta}(t)$, of the lift signal. 3. The EKF frequency estimate is used to tune the filter to avoid introducing phase lag. 4. Finally, the phase estimate $\hat{\theta}(t)$ from EKF is used to march along $\phi_{\text{optimal}}$.

Feedback control of the optimized waveform was able to reproduce the high-lift limit cycle from the optimization, but starting from an arbitrary phase of the baseline limit cycle. Also, it allowed us to phase lock an essentially arbitrary waveform, thus enabling us to investigate the sensitivity of the flow to the phase shift and other features of the optimized waveform. By using fewer harmonics in the Fourier expansion of the optimized waveform, we demonstrated that smoothing the dip between the two pulses has little impact on the lift performance; this characteristic is more of an energy-saving feature. We also showed that the phase-locked limit cycle with optimized waveform was sensitive to changes in the phase shift, $\theta_{\text{desired}}$ from $0^\circ$. 

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The sharp decrease in lift performance with negative phase shift is due to the steep drop in the optimized waveform after its peak. The lift penalty with positive phase shift indicates that the forcing is less effective after the maximum lift has occurred.

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