A study of linear wavepacket models for subsonic turbulent jets using local eigenmode decomposition of PIV data

Daniel Rodríguez a,*, André V.G. Cavalieri b, Tim Colonius c, Peter Jordan d

a School of Aeronautics, Universidad Politécnica de Madrid, Spain
b Divisão de Engenharia Aeronáutica, Instituto Tecnológico de Aeronáutica, São José dos Campos, SP, Brazil
c Department of Mechanical Engineering, California Institute of Technology, Pasadena, CA, USA
d Département de Fluides, Thermique, Combustion, Institut PPrime, CNRS-Université de Poitiers-ENSMA, Poitiers, France

ARTICLE INFO

Article history:
Available online 22 March 2014

Keywords:
Turbulent jets
Stability of nonparallel flow
Normal modes

ABSTRACT

Locally-parallel linear stability theory (LST) of jet velocity profiles is revisited to study the evolution of the wavepackets and the manner in which the parabolized stability equations (PSE) approach models them. An adjoint-based eigenmode decomposition technique is used to project cross-sectional velocity profiles measured using time-resolved particle image velocimetry (PIV) on the different families of eigenmodes present in the LST eigenspectrum. Attention is focused on the evolution of the Kelvin–Helmholtz (K–H) eigenmode and the projection of experimental fluctuations on it, since in subsonic jets the inflectional K–H instability is the only possible mechanism for linear amplification of the large-scale fluctuations, and governs the wavepacket evolution. Comparisons of the fluctuations extracted by projection onto K–H eigenmode with PSE solutions and PIV measurements are made. We show that the jet can be divided into three main regions, classified with respect to the LST eigenspectrum. Near the jet exit, there is significant amplification of the K–H mode; the PSE solution is shown to comprise almost exclusively the K–H mode, and the agreement with experiments shows that the evolution of this mode dominates the near-nozzle fluctuations. For downstream positions, the Kelvin–Helmholtz mode becomes stable and eventually merges with other branches of the eigenspectrum. The comparison between PSE, experiment and the projection onto the K–H mode for downstream positions suggests that the mechanism of saturation and decay of wavepackets is related to a combination of several marginally stable modes, which is reasonably well modeled by linear PSE, but cannot be obtained in the usual application of locally-parallel stability dealing exclusively with the K–H mode. In addition, the projection of empirical data on the K–H eigenmode at a near-nozzle cross-section is shown to be a well-founded method for the determination of the amplitudes of the linear wavepacket models.

© 2014 Elsevier Masson SAS. All rights reserved.

1. Introduction

Reducing the noise radiated by turbulent jets is a technological problem of great importance that has been the subject of continued research since the appearance of commercial jet airliners. Improvements like the introduction of turbofans and subsequent increase in bypass ratio, that reduced remarkably the noise emitted by subsonic commercial jets, were based on Lighthill’s theory of aerodynamic noise generated by turbulence [1] that states that

the sound intensity emitted by jet turbulence, thought at the time to consist solely of compact eddies, is proportional to \(U_j^8\), where \(U_j\) is the jet exit velocity. More recent efforts directed at jet noise reduction consider passive and active control methods that require a deeper understanding of the underlying physics and simple models for the quantitative prediction of the emitted noise. Over the last decades, a theory based on the existence and dynamics of intermittent, large-scale turbulent structures as the prominent noise sources has been elaborated, and extensive comparison with data from simulations and experiments demonstrate its utility. Statistical descriptions of these large-scale structures, coherent along several diameters in the axial direction, resemble instability waves and are commonly referred to as wavepackets. Jordan and Colonius [2] review the theoretical framework and supporting experimental evidence of wavepackets in jets.

* Corresponding author. Tel.: +34 649201093.
E-mail addresses: dani@torroja.dmt.upm.es, dani@torroja.dmt.upm.es (D. Rodríguez).
http://dx.doi.org/10.1016/j.euromechflu.2014.03.004
0997-7546/© 2014 Elsevier Masson SAS. All rights reserved.
Following the experimental observations of wavepackets in turbulent jets [3,4], attempts have been made to model them as instability waves or perturbations of the turbulent mean flow [3,5]. Crighton and Gaster [6] postulated that linear stability theory (LST) is appropriate for the modeling of wavepackets, provided that the large-scale structures constitute only a small fraction of the total kinetic energy and that most of the nonlinearities take place in the establishment of the turbulent mean flow, and are therefore already accounted for with its use as the reference state. While a rigorous analysis proving this hypothesis is still lacking, the good agreement between experiments and linear wavepacket models serves as a posteriori proof of their validity. One possible explanation for the success of linear instability wavepacket models based on the turbulent mean flow is the relative fast spread of the mixing layer in the turbulent mean flow, as compared with a steady laminar shear layer. The amplification, saturation and decay of the instability waves on a turbulent mean flow seem to be governed by the spread of the annular mixing layer and not by nonlinear interactions between the coherent structures corresponding to different frequencies.

In order to take into account the divergence of the mean flow along the axial direction, multiple-scale analysis [6–8] and parabolized stability equations (PSE) [9–13] have been used. Both approaches are based on the locally-parallel linear stability theory (LST), but assume a mild variation of the properties of the mean flow along the dominant streamwise direction and introduce higher-order corrections to the perturbations on account of this variation. Recent advances in computational power have enabled the study of modal perturbations without any assumption on the properties of the mean flow or the perturbations [14], but the computational expense of this analysis would render prohibitive the use of these wavepacket models as predictive tools for design purposes.

Most of the studies comparing wavepackets in turbulent jets with stability theory considered forced jets, for which the introduction of controllable phase-locked disturbances enables a direct comparison between experiments and theoretical models; the wavepackets predicted by LST were found to be in good agreement with the measured near-field fluctuations [3,15–17]. In the case of natural or unforced jets, the lack of a phase reference precluded achieving satisfactory quantitative comparisons between theoretical wavepackets and the experimental data until relatively recently. Suzuki and Colonius [18] considered a series of measurements performed at the NASA Glenn SHJAR facility [19], in which a microphone phased-array was carefully placed just outside of the turbulent mixing layer, in a region where pressure fluctuations are expected to be mostly related to the large-scale structures in the jet. This experimental technique, almost identical to the one used by Tinney and Jordan [20] and more recently by Breakey et al. [21] for the same jet configuration studied in this work, was instrumental in the successful eduction of the near-field pressure fluctuations corresponding to the coherent structures, leading to remarkable comparisons with the calculated linear wavepackets.

The identification of wavepackets in the velocity field presents additional difficulties. When considering natural jets, the signature of the azimuthally-coherent wavepackets is clearer in the near pressure field, whose structure is considerably simpler than that of the turbulent velocity field: the energy of the hydrodynamic pressure field is concentrated in the few lower-order azimuthal modes, in clear contrast to the velocity field [13,22]. On the other hand, the wavepacket signature in the velocity field of forced turbulent and transitional jets is relatively clear [4,23]. Its eduction in natural jets is more difficult due to the lack of a phase reference, as mentioned previously, but also to the smaller energy content of the coherent structures compared to the total velocity fluctuations. Recently, Kerhervé et al. [24] employed linear stochastic estimation to deduce the velocity field of wavepackets corresponding to the noise radiated to the far field in a subsonic jet. Gudmundsson and Colonius [12] compared the velocity field of unforced jets measured using particle image velocimetry (PIV) in cross sections with the results of linear PSE showing an encouraging agreement. Cavaliere et al. [13] pursued further this issue by combining hot-wire anemometry and stereoscopic time-resolved PIV (TR-PIV) in cross-stream planes for measuring the velocity field and a microphone ring array at a polar angle of $\theta = 20^\circ$ for the sound radiation. Correlations around 10% were found between the velocity and the acoustic pressure signals, once decomposed into azimuthal Fourier modes, this being significantly higher than those reported previously for two-point flow-acoustic correlations. This result suggests that, despite the broadband character of turbulent structures in the velocity field in which higher azimuthal modes dominate, velocity fluctuations exist at the lowest azimuthal modes that are intimately related to the noise emitted to the far-field. In the same work, the axisymmetric and first helical modes of the velocity field were compared with PSE wavepackets. For all but the lowest frequencies considered, close agreement was found for all velocity components in the spatial amplification, up to the end of the potential core.

In the present work we revisit the velocity field associated with the large-scale structures present in subsonic turbulent jet flows. A detailed study of the properties of the parallel-flow LST eigenspectrum and its evolution along the axial direction is performed. Definition of an adjoint LST problem and a corresponding bi-orthogonality relation allows us to project fluctuation profiles from experimental measurements or high-fidelity numerical simulations on the different families of eigenmodes, in order to obtain their separate contributions to the total fluctuation field, and investigate which modes are relevant for each region of wavepackets within jets. Similar eigenmode decompositions have been performed in the past in the study of laminar and transitional flows using data from experiments or numerical simulations [25,26]. Rodríguez et al. [27] presented an eigenmode decomposition of LES data for supersonic turbulent jets, but an analogous study employing experimental data for turbulent jets had not been attempted before. In this paper, the cross-sectional velocity profiles measured using TR-PIV from Cavaliere et al. [13] are used in a bi-orthogonal eigenmode projection. Comparisons of the fluctuations extracted as the projection of the experimental data on the eigenmode corresponding to the dominant Kelvin–Helmholtz instability with PSE wavepacket models and the experimental velocity profiles are done with the aim of getting a deeper insight into the relation between the physical mechanisms that underpin wavepackets and their modeling using PSE.

The remainder of the paper is organized as follows. Section 2 describes the jet configuration, experimental set up and data postprocessing, while Section 3 summarizes the theoretical background of the wavepacket model using LST and PSE. The different families of eigenmodes in the LST eigenspectrum and their evolution with the spread of the mean jet mixing layer are studied in Section 4. The projection of the experimental velocity profiles on the LST eigenmodes is done in Section 5, and comparisons of the jet centerline velocity from the projection is compared with hot-wire and TR-PIV measurements. Comparisons of the cross-sectional velocity profiles are done in Section 6. Section 7 discusses the relation between the qualitative changes in the LST eigenspectrum along the axial direction and the agreement in the comparisons between the PSE wavepacket model and the experimental data. The main conclusions are summarized in Section 8.

2. Flow configuration and experimental setup

An isothermal subsonic round jet with exhaust Mach number $M_e = 0.4$ and the Reynolds number based on the exit velocity
U_j and nozzle diameter D equal to Re = 4.2 \times 10^3 is considered here. This flow configuration was studied in a series of experiments performed in the ‘Bruit et Vent’ anechoic facility at the Centre d’Études Aérodynamiques et Thermiques (CEAT), Institut Pprime, Poitiers, France. Different aspects of the wavepackets have already been addressed using this experimental set up [13,28]. A detailed presentation of the experimental facility, measurement equipment and data processing can be found in the given references; only the information most relevant to the present work, related to the postprocessing of experimental data, is repeated in what follows.

A convergent section was located upstream of the jet exit with an area contraction of 31, followed by a straight circular section of length 150 mm and diameter D equal to 0.05 m. A carborundum trip placed 135 mm (2D) upstream of the nozzle exit was used to ensure transition of the inner boundary layer before the lip, ensuring that the annular jet mixing-layer is initially turbulent [13]. The boundary-layer displacement thickness measured using hot-wire anemometry at the nozzle exit was \( \delta_{bl} = 4.5 \) mm and the associated Reynolds number \( Re_{bl} \) was \( 3.8 \times 10^4 \).

Velocity measurements were obtained using a traversing single hot-wire and with stereoscopic, time-resolved particle image velocimetry (TR-PIV). Wires with diameter of 2.5 \( \mu \)m and length of 0.7 mm (about 0.15\x) were used with a Dantec 55M01 anemometer. The corner frequency of this setup was 30 kHz, corresponding to a Strouhal number \( St = fD/U_j \approx 7 \), sufficiently high to avoid significant aliasing effects in the low-frequency, energy containing part of the hot-wire spectra. Calculations of power spectra were performed by averaging 400 blocks of data without overlap, with frequency resolution corresponding to \( \Delta St = 0.022 \).

Stereoscopic, TR-PIV measurements were performed at different cross-planes. The sampling frequency was 5 kHz, corresponding to \( St = 1.82 \); this frequency was the maximum possible value with the available equipment. Details of the evaluation of possible errors due to aliasing can be found in Cavaleri et al. [13]. In the present study we focus on the analysis of the jet measurements on the Strouhal number range of 0.1 \( \leq St \leq 0.9 \) to diminish significant errors due to aliasing, but some noticeable effects can still be observed at the higher frequencies. A total of 19414 image pairs were recorded. Image-processing consisted of a five-pass correlation routine with 64 \times 64 pixel correlation overlap at each pass, done with LaVision software DaVis 8. This leads to velocity fields with 114 \times 102 velocity vectors. This grid was subsequently interpolated to cylindrical coordinates to allow the expansion of the velocity field in azimuthal Fourier modes. Frequency spectra were calculated using block lengths corresponding to a resolution in Strouhal number of \( \Delta St = 0.025 \), resulting in 523 blocks with 50% overlap. During the first phase of analysis of the results presented herein, it was found that 100 blocks suffice to attain convergence of the statistical velocity profiles, and only 100 blocks were used subsequently. All power spectral densities presented in this work are single-sided.

The hot-wire measurements were used to determine the mean axial velocity field. To avoid numerical issues in the subsequent numerical computations stemming from residual non-smoothness in the measured field, the mean flow is fitted with a Gaussian profile:

\[
\bar{u}_a(x)/U_j = \begin{cases} 
\frac{1}{\sigma_a(x)} \exp\left(-\left(r - R(x)\right)^2/\sigma^2(x)\right) & \text{if } r < R(x) \\
0 & \text{otherwise.}
\end{cases}
\]  

(1)

The parameters \( u_a \), \( R \) and \( \sigma \) are determined by a least-squares fitting of the experimental data. A zero mean pressure gradient is assumed, and the Crocco–Busemann relation particularized for the isothermal jet

\[
\frac{\bar{T}}{T_\infty} = \left( M_j - \bar{u}_a/\bar{a}_\infty \right) \bar{u}_a/(2\alpha_\infty) + 1/(\gamma - 1), \quad \gamma = 1.4
\]  

(2)

where \( \bar{a}_\infty \) is the ambient speed of sound, is used to determine the mean temperature and density fields. Spatial variations in the mean temperature relative to the mean value are small (at most, \( O(M_j^2(\gamma - 1)/2 \approx 0.032 \)). The radial velocity field is then obtained from the continuity equation. Fig. 1 shows the main features of the jet mean velocity field.

Three distinct spatial regions exist in the development of the mean turbulent jet [29], that can be described quantitatively using the analytic profile (1). Immediately downstream of the nozzle lip, a thin (but turbulent) mixing-layer exists surrounding a well-defined potential core. Top-hat velocity profiles are excellent approximations of this region (\( \delta \ll D, R \approx 0.5D \)) and theoretical considerations based on self-similarity of the velocity profile [30] show that \( \bar{u}_c \) is constant and \( \delta \) grows linearly with \( x \). Another region of self-similarity starts downstream of the potential core, and corresponds to a fully developed turbulent jet (\( \delta \sim D, R \approx 0 \)). In this region, \( \bar{u}_c \) decays approximately as \( \sim 1/x \) and \( \delta \) grows linearly with \( x \) [30]. Between the two spatial regions of self-similarity lies a transition region, in which a narrow potential core still exists but \( \delta \sim D \).

3. Theoretical background

This section summarizes the approach following for the modeling of the wavepackets in natural turbulent jets. A broader discussion of the approach and numerical implementation can be found elsewhere [12,27,31]. The turbulent flow field is decomposed into a time-averaged (or mean) flow and fluctuations, \( \bar{\mathbf{q}}(x, t) = \mathbf{q}(x) + \mathbf{q}'(x, t) \). A cylindrical coordinate system is used where \( x = (x, r, \theta) \) are respectively the axial, radial and azimuthal coordinates. The vector of fluid variables is denoted by \( \mathbf{q} = [u_x, u_r, u_\theta, p, \xi] \), where \( u_x, u_r \) and \( u_\theta \) are the axial, radial and azimuthal velocity components, \( p \) is the pressure and \( \xi \) is the specific volume, inverse of the density. Fourier modes are introduced for frequency \( \omega = 2\pi M_j St \) and wavenumber \( m \) following

\[
\mathbf{q}'(x, t) = \sum_\omega \sum_m q_{om}(x, r) e^{im\theta} e^{-i\omega t}.
\]  

(3)

Note that a linear relation exists between \( \omega_q \) and the Strouhal number, \( St = M_j \), stands for the Mach number at the jet exit. The Fourier-transformed perturbation component in \( \omega \) and \( m \) is referred to as \( q_{om} \). The two dimensionless frequencies will be used indifferently along the paper.

3.1. Parabolized stability equations

Parabolized stability equations (PSE) [32] are an evolution of the multiple-scale approach that has been employed to model the wavepackets owing to the slowly divergent nature of the jet mean flow along the axial direction [9,10,12,13]. In PSE, the fluctuations \( q_{om} \) are decomposed into a slowly-varying shape function \( \tilde{q}_{om} \), evolving in the same scale as the mean flow, and a rapidly varying wave-like part:

\[
\tilde{q}_{om}(x, r) = \tilde{A}_{om}(x) \tilde{q}_{om}(x, r)
\]  

(4)

The axial wavenumber \( \alpha_{om} = \alpha_r + i\alpha_\theta \) is a complex quantity for which a slow variation is also assumed. The coordinate \( x_0 \) is the axial location where the PSE integration is initialized, typically a cross-section close to the nozzle lip. Introducing this decomposition into the compressible Navier–Stokes, continuity and energy equations and subtracting the terms corresponding to the mean
flow we arrive at the system of equations
\begin{equation}
\begin{pmatrix}
    A + B \frac{d\alpha}{dx} + C \frac{\partial}{\partial x} + D \frac{\partial}{\partial r} + E \frac{\partial^2}{\partial r^2} + F \frac{\partial^2}{\partial x \partial r}
\end{pmatrix} \tilde{q}_{\text{om}}(x, r) = R_{\text{om}}.
\end{equation}

The linear operators A to F depend on the mean flow quantities, the Reynolds number, Mach number, frequency $\omega$, azimuthal wavenumber $m$ and axial wavenumber $\alpha$. Details of the derivation of Eq. (5) can be found elsewhere [12,31,33]. The system of Eq. (5) is a linear spatial operator for each mode $(\omega, m)$ with the forcing term $R_{\text{om}}$ accounting for the mode-dependent Reynolds stresses. For unforced turbulent jets, the small relative amplitude of the individual modes suggests that nonlinear interactions between the lower modes can be neglected, as most nonlinear effects are implicit in the mean flow, so that $R_{\text{om}} \approx 0$ [6,16].

After spatial discretization of the radial direction and imposition of adequate boundary conditions, the system (5) takes the form
\begin{equation}
L \frac{\partial \tilde{q}_{\text{om}}}{\partial x} = R \tilde{q}_{\text{om}}
\end{equation}
where $L = C + F \mathcal{D}_r$ and $R = -(A + B dx/\partial x + D \mathcal{D}_r + E \mathcal{D}_r^2)$ have been introduced. The matrix operators $\mathcal{D}_r$ and $\mathcal{D}_r^2$ stand for the discrete versions of the first and second order spatial differentiation along the radial direction, respectively. The system of equations is discretized using fourth-order central finite differences in the radial direction, closing the domain with the characteristic boundary conditions of Thompson [34]. The boundary conditions at the centerline are derived following Mohseni and Colonius [35]. The computational domain extends up to 5D in the radial direction, and the number of discretization points was checked to be sufficient to converge the results. In particular, 301 points were used to have grid-independent PSE solutions for all the frequencies considered, as well as convergence of the first 4–5 decimal places of the Kelvin–Helmholtz eigenmode in the LST analyses described in Section 3.2.

The decomposition of (4) is ambiguous in that the spatial growth can be absorbed into the shape function $\tilde{q}_{\text{om}}$ or the complex amplitude $A_{\text{om}}$. Following Herbert [32], the normalization condition
\begin{equation}
\int_{0}^{\infty} \tilde{u}^* \frac{\partial \tilde{u}_{\text{om}}}{\partial x} r dr = 0,
\end{equation}
where $\tilde{u}_{\text{om}}$ refers to the vector of three velocity components and $*$ denotes complex conjugation, is imposed individually to every $(\omega m)$ mode, removing the exponential dependence from $\tilde{q}_{\text{om}}$.

Expression (6) is an initial value problem with the axial distance $x$ acting as the marching direction. It requires initial conditions for the perturbation shape $\tilde{q}_{\text{om}}$ and wavenumber $\tilde{\alpha}_{\text{om}}$ at the inlet location $x_0$. The amplitude factor $A_{\text{om}}$ can be determined a posteriori, as linear PSE results are independent of amplitude. Information provided by locally-parallel linear stability theory (LST) is usually employed in the determination of the inlet conditions.

### 3.2. Local stability eigenvalue problem

The local stability eigenvalue problem (EVP) is derived here from the PSE approximation, by assuming $d\alpha_{\text{om}}/dx \approx 0$ and $\partial \tilde{\alpha}_{\text{om}}/\partial x \approx \omega_{\text{om}} \tilde{q}_{\text{om}}$. These assumptions are the usual ones in the derivation of local stability problem of the Orr–Sommerfeld kind, but one difference exists in the present approach: the second axial derivatives of the perturbations appearing in the viscous term are neglected here, consistently to what is done in PSE. These terms yield $\alpha^2$ terms in the usual spatial stability problem, requiring a special treatment to recast the equations as a linear EVP. Neglecting these terms alters the results from LST by eliminating two branches of upstream propagating vortical and entropic waves in the eigenspectrum [36], while leaving the other families practically unaltered. In particular, the effect of these terms in the discrete Kelvin–Helmholtz eigenmode is very small, as the neglected terms are proportional to $Re^{-1} \approx 10^{-5}$.

From the approximations above, the following matrix EVP results:
\begin{equation}
\alpha L \tilde{q} = R \tilde{q},
\end{equation}
where the subscripts $(\omega m)$ have dropped for simplicity. Operators $L$ and $R$ are those in (6), but modified for $\alpha = 0$ [27] and with
the non-parallel terms in the mean flow neglected. The eigenvalue problem [8] describes the spatial growth or decay of disturbance waves with a fixed real frequency $\omega$ and azimuthal wavenumber $m$.

The solution of (8) delivers a complete eigenspectrum containing discrete and continuous solutions, but the spatial discretization of the problem leads to a discretization of the continuous branches. The eigenvalue and eigenfunctions corresponding to the discrete eigenmode $n$ will be referred to as $\alpha_n$ and $\hat{q}_n$, in what follows. The properties of the LST eigenspectrum will be discussed in Section 4.

3.3. Bi-orthogonal decomposition and adjoint LST problem

As was demonstrated for a flat-plate boundary layer, the eigenspectrum from LST is a complete set and an arbitrary perturbation profile can be decomposed exactly as a linear superposition of the eigenfunctions $\hat{q}_n(r) = \sum a_n \hat{q}_n(r)$ [37,38]. Due to the nonnormality of the linearized Navier–Stokes equations, the eigenfunctions do not comprise an orthogonal system, and the solutions of the adjoint problem are required in order to obtain a bi-orthogonality relation. The adjoint eigenfunction corresponding to the discrete eigenmode $n$ is denoted by $\hat{q}_n^\alpha$. By construction of the adjoint EVP [27], a bi-orthogonality relation holds between the direct and adjoint eigenfunctions:

$$(\alpha_n - \hat{\alpha}_n) \left( \hat{q}_n^\alpha \right)^\dagger L \hat{q}_n = 0,$$

where $H$ denotes Hermitian transposed. Relation (9) can be used to obtain the weighting coefficients $a_n$ in the projection of an arbitrary perturbation profile $q(r)$:

$$a_n = \left( \hat{q}_n^\alpha \right)^\dagger L \hat{q}_n / \left( (\hat{q}_n^\alpha)^\dagger L \hat{q}_n \right).$$

The adjoint eigenmodes illustrate the spatial regions where the individual eigenmodes are more receptive to the introduction of disturbances or external forcing. As the adjoint and direct eigenfunctions usually differ notably in regions of mean flow shear due to nonnormality [39], the regions of highest receptivity are not necessarily those in which the modal perturbation attains its highest amplitude. This result has important implications when considering the initial conditions in evolution problems or the location of actuators for flow control [40].

In this work, the adjoint eigenfunctions are used as a filter of the relative contribution of the individual eigenmodes onto the arbitrary perturbation profile [41,42]. This decomposition represents a theoretically-founded method for the determination of the amplitude associated with the Kelvin–Helmholtz eigenmode at the different measured cross-sections. The bi-orthogonal projection of empirical data (from experiments or high-fidelity numerical simulations) at a single cross-section in the vicinity of the nozzle exit has recently been shown to be a valid method for the amplitude calibration of linear wavepacket models [27,43].

4. Properties of the eigenspectrum

The solution of the locally-parallel linear stability EVP (8) delivers a complete eigenspectrum of waves. Several studies in the past addressed the eigenspectrum of jet-related velocity profiles, either in the compressible inviscid [44,45], viscous incompressible [29] or viscous compressible limits [27,46]. Most of these studies are focused on the instability of top-hat shaped velocity profiles, representative of the first diameters in the axial direction. However, due to the spread of the mean mixing-layer and the closure of the potential core, the properties of the eigenspectrum change significantly along the axial direction.

An important parameter in the analysis of mixing-layer instability relates the frequency $\omega$ to the width of the layer, $\delta$ [47]; the EVP (8) can then be rewritten for the rescaled frequency $\omega \delta / D$. The qualitative differences existing in the mean velocity profiles yield important changes in the LST eigenspectrum for each frequency, that also impact the axial evolution of the PSE solutions. Three spatial regions can be defined then, according to the value of $\omega \delta / D$. Fig. 2 shows representative eigenspectra of the three spatial regions. Some eigenfunctions corresponding to the different eigenmode branches are shown in Fig. 3.

4.1. Region I: annular mixing-layer, $\omega \delta / D < 1$

Region I is characterized by a thin mixing-layer ($\delta \ll 1$) and $R \approx 0.5D$. This translates into $\omega \delta / D < 1$ for the range of frequencies of interest. The top-hat velocity profile typical of this region presents three distinct spatial zones along the radial direction: a well-defined potential core, a thin annular mixing-layer and the outer flow, i.e., the uniform unbounded stream surrounding the jet. The three different zones are readily identified in Fig. 3(a). Fig. 2(a) depicts a representative eigenspectrum of this region.

The outer flow supports a continuous spectrum of acoustic waves [8]. The acoustic waves are of inviscid nature and correspond to solutions of the Helmholtz equation. For quiescent free-stream or stream moving with subsonic velocity, the acoustic branches comprise a finite range of wavenumbers corresponding to waves propagating in the radial direction and an infinite range of wavenumbers corresponding to evanescent waves [45] (filled squares in Fig. 2).

Two additional branches present in the computed eigenspectra (open squares in Fig. 2) are also classified as acoustic waves. These branches are physically meaningful for $m \neq 0$. In the present computations the momentum conservation equation in the azimuthal direction is maintained also for $m = 0$ permitting the appearance of these branches, but it is decoupled from the others and the branches correspond to spurious modes, being irrelevant for the present application.

The potential core can be regarded as a bounded uniform flow, supporting infinite discrete eigenmodes equivalent to the continuous spectra. Of particular significance in the present work are the
branches of vorticity and entropy waves, of viscous origin. These waves are stable and propagate either upstream or downstream with phase velocity equal to the mean velocity in the potential core, i.e., to the jet exit velocity. The approximation of neglecting the second-order axial derivative terms in the EVP neglects the upstream propagating branches \([36]\) and consequently they are not recovered in the present eigenspectrum. The eigenmodes corresponding to the vorticity and entropy branches obtained in the present EVP are denoted by asterisks in Fig. 2.

Finally, the inflectional mixing-layer supports the discrete eigenmode corresponding to the Kelvin–Helmholtz (K–H) instability mechanism, denoted by a filled circle in the figures. For the velocity profiles typical of most round subsonic jets, K–H is the only potentially unstable eigenmode. The K–H eigenmode is always unstable over Region I for the frequency range \(0.1 \leq \text{St} \leq 0.9\) studied in the present work.

4.2. Region II: transition, \(\omega \delta / D \sim 1\), small \(R\)

Region II is characterized by a relatively thick mixing layer \(\langle \delta \sim D \rangle\), comparable to the size of the potential core and the jet diameter, that leads to qualitative changes in the inflectional instability waves. In region II, \(\omega \delta / D\) takes values of magnitude \(O(1)\) (between 0.1 and 1.7), depending on the frequency and axial location. An eigenspectrum representative of this region is shown in Fig. 2(b), corresponding to \(x = 3D\) and \(\text{St} = 0.5\) \((\omega \delta / D \approx 0.44)\).

When \(\omega \delta / D \sim 1\), the mixing-layer is relatively thick and permits the existence of more than a single discrete eigenmode associated with it: a new branch of discrete mixing-layer eigenmodes appears in the eigenspectrum (denoted by the open circles close to the branch of core eigenmodes). When the rescaled frequency is small, one eigenmode – the Kelvin–Helmholtz one – is unstable, and clearly distinguishable from the others (Fig. 3(b)). As \(\omega \delta / D\) increases, the K–H eigenmode first becomes stable and then eventually becomes part of the branch. The eigenfunctions corresponding to this branch e.g., Fig. 3(c)–(e)) show localized oscillations in the mixing layer; as for the other branches, the eigenmodes can be enumerated following the number of peaks or nodes that the eigenfunction presents within this region.

The description of branches of vortical and entropic waves within the potential core also changes with increasing axial coordinate. As the mean velocity distribution diverges from the uniform flow, the branch is distorted from the nearly vertical line to a line tilted towards higher real wavenumbers, or equivalently,
lower phase speeds. The eigenfunctions corresponding to the first two core vorticity modes are shown in Fig. 3(f) and (g). The outerfield acoustic branches remain nearly unaltered along the axial direction, as the spread of the mixing region does not modify substantially the free-stream supporting them. Fig. 3(h) and (i) depict typical acoustic eigenfunctions.

4.3. Region III: developed jet, \( \omega \delta / D > 1, R \approx 0 \)

The precise location where Region III starts depends on the frequency, but the end of the potential core (that occurs at \( x \approx 6D \)) delimits the extension of Region II even for the lowest frequencies. The mean velocity profile presents two zones with a smooth transition: the outer field and a thick mixing layer extending from the jet axis. The branches associated with the potential core and the mixing-layer eigenmodes coalesce into a single group of eigenmodes, with amplitude functions as those shown by Fig. 3 in Morris [29] and Fig. 3(f) and (g); these eigenfunctions are qualitatively identical to those of the core vorticity branch in regions I and II. Again, the changes in the mean profile in this region do not alter significantly the eigenmodes in the acoustic branches. Fig. 2(c) shows the eigenspectrum corresponding to \( x = 6D \) and \( St = 0.5 \).

5. Bi-orthogonal projection of the TR-PIV data

The bi-orthogonal projection discussed in Section 3.3 can be used to decompose the experimental data into separate contributions of the different eigenmodes from linear stability theory. In this section, the projection is employed to extract, cross-section by cross-section, the fluctuation component associated with the Kelvin–Helmholtz eigenmode alone.

5.1. Approximate projection using experimental data

The determination of the weighting coefficients in the eigenmode decomposition (10) requires the evaluation of the inner product between the adjoint eigenfunction and the total fluctuation profile \( \tilde{q}_k \) to be decomposed, that comprises all the five fluid variables in the problem \( u_x, u_y, u_z, p \) and \( \zeta \). When numerical simulation data are used, all the fluid variables are available and the projection can be done in an exact manner [27]. When experimental data is used, only some of the variables are known. The present TR-PIV data provides the three velocity components but no information on the pressure or specific volume. Some works exist in the literature [42], in which additional information, e.g. the absence of contribution from some families of eigenmodes, is exploited in order to close the problem as an algebraic system of equations for the weighting amplitudes. This kind of assumption is not possible for the present turbulent flow, as all the different families are expected to contribute to some extent. In this work, the contributions of the (unknown) pressure and specific volume fluctuations to the inner product are neglected. The relative importance of these variables is hypothetized to be small for the isothermal and low Mach number jet under consideration, as they account mostly to compressibility effects; the inner product corresponding to the analogous biorthogonal projection in incompressible flow only comprises the velocity components.

A second problem associated with the use of experimental data concerns the limited spatial resolution and domain size that can be probed in the experiments. Due to the power limitation in the laser used, the time-resolved PIV setup used interrogates a relatively small spatial window on each plane; the data is truncated at a radius ranging between 0.66D and 0.82D, depending on the axial location. While this extension is sufficient to contain the peak turbulent fluctuations, concentrated at the lipline, it may be too short for the accurate evaluation of (10).

<table>
<thead>
<tr>
<th>St</th>
<th>Exact</th>
<th>Approx.</th>
<th>Truncated</th>
<th>Approx. and truncated</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_4 )</td>
<td>( C_6 )</td>
<td>( e(%) )</td>
<td>( C_4 )</td>
<td>( C_6 )</td>
</tr>
<tr>
<td>0.1</td>
<td>1.6817</td>
<td>1.7766</td>
<td>5.3</td>
<td>1.6299</td>
</tr>
<tr>
<td>0.2</td>
<td>1.6282</td>
<td>1.7231</td>
<td>5.5</td>
<td>1.5083</td>
</tr>
<tr>
<td>0.3</td>
<td>1.2346</td>
<td>1.1952</td>
<td>3.3</td>
<td>1.2650</td>
</tr>
<tr>
<td>0.4</td>
<td>1.2700</td>
<td>1.1662</td>
<td>8.9</td>
<td>1.3333</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1692</td>
<td>0.1547</td>
<td>9.4</td>
<td>0.1740</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1353</td>
<td>0.1278</td>
<td>5.8</td>
<td>0.1365</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1477</td>
<td>0.1420</td>
<td>4.0</td>
<td>0.1481</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0260</td>
<td>0.0255</td>
<td>1.8</td>
<td>0.0260</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1950</td>
<td>0.1905</td>
<td>2.4</td>
<td>0.1951</td>
</tr>
</tbody>
</table>

In order to have a quantitative estimate of the error due to the approximations made, the cofactors \( C_k \) with and without neglecting the pressure and specific volume variables (the former case referred to as approximated), and two truncations in the radial direction: \( r = 5D \) corresponding to the domain boundary in the LST computations, and \( r = 0.66D \), corresponding to the minimum extent of the TR-PIV data. Table 1 shows the cofactors and the relative errors for different \( St \) for \( m = 0, at x = 4D \). Similar values are obtained at other axial locations. For the first helical mode the errors are found to be slightly lower. In view of Table 1, the relative errors in the projected amplitudes are estimated to be less than 10% for the K–H eigenmode. The relative eigenfunctions for different eigenmode families exhibit different decay rates on the radial direction, indicating different sensitivities to the data truncation. As an example, the same test for eigenmodes corresponding to free-stream acoustic wave yields errors of the same order of the cofactor, invalidating the projection.

5.2. Projection on the Kelvin–Helmholtz eigenmode

The contribution of the K–H eigenmode from LST at each cross section is computed by means of projecting the TR-PIV velocity profiles on its adjoint eigenfunction. Along with the Fourier transform in the azimuthal direction, a finite-time Fourier transform is applied to the experimental data, resulting in a set of realizations, or Fourier-transform of data blocks. This procedure is one standard step in the computation of power spectra, and the same caution regarding the choice of frequency bin \( \Delta St \) and the highest reliable frequency apply. The post-processing of TR-PIV data was explained in Section 2.

The fluctuation profiles corresponding to different data blocks may vary notably along the total time of the experiment, having a strong impact on the projection amplitude. On the other hand, using the measured profiles averaged over blocks is not recommended as the phase relationships may not be adequately preserved in the averaging process. Instead, each Fourier-transformed block is projected on the K–H eigenmode and the resulting amplitudes are then averaged, enabling direct comparisons between the measured velocity power spectral density and the extracted K–H eigenmode.

Fig. 4 compares the centerline axial velocity fluctuations for \( m = 0 \) as measured using a traverse hot-wire, extracted from the TR-PIV measurements, and its projection on the K–H eigenmode. The symmetries of the problem imply that only the axial velocity component of the \( m = 0 \) Fourier mode contributes to the centerline hot-wire measurements in a significant manner. The variance of the data projection is also shown to illustrate the variability of the amplitudes over blocks. Some differences are observed between the hot-wire and TR-PIV measurements. Due to
the limitation in frequency imposed by the laser device, the TR-PIV sampling frequency was only \( St = 1.82 \), resulting into higher spectral amplitudes using TR-PIV due to aliasing, as observed for the higher frequencies. A different kind of discrepancy is found up to \( x \approx 2D \), where TR-PIV presents higher amplitudes than hot-wire measurements for all frequencies. This is attributed to different signal-to-noise ratios between TR-PIV and hot-wire measurements. Hot wires are more appropriate for the measurement of small velocity fluctuations, such as the ones inside the potential core of the jet for low \( x/D \). A detailed study of the effect of aliasing in the TR-PIV data was performed in Cavalieri et al. [13]. It was found that time-resolved PIV measured significantly higher amplitudes of the velocity fluctuations inside the core; however, taking the first mode from a proper orthogonal decomposition (POD) of the measurements filters the uncorrelated noise, leading to amplitudes which are close to the hot-wire results, as was shown in Cavalieri et al. [13] and also in Figs. 5–8.

The extracted K–H eigenmode is found to match approximately the evolution of the hot-wire measurements over the first diameters, both in absolute amplitude and spatial growth. It should be remarked that the K–H amplitudes are obtained from the TR-PIV data and not from the hot-wire measurements, and consequently are also affected by aliasing. The agreement between the projected K–H eigenmode and hot-wire confirms that the discrepancies observed between TR-PIV and hot-wire measurements are due to the mentioned uncorrelated noise in the TR-PIV data at the axis. The adjoint K–H eigenfunctions (not shown in this paper) peak at the lipline and decay very fast towards the jet axis, so either physical or unphysical contamination at the axis has a very limited contribution to the projection amplitude and are filtered out.

The close match between the projected K–H eigenmode and the hot-wire measurements extends for a distance of few diameters along the axial direction, the extent of which depends strongly on the frequency. The agreement is lost downstream and then the projection on the K–H eigenmode overestimates the centerline velocity. The spatial growth of the extracted K–H eigenmode in Fig. 4 is not predicted by LST, but is imposed by the increase in the averaged projection amplitude that only considers the shape of the fluctuation profiles. Consequently, a good agreement in the projected amplitude implies a good agreement between the measured velocity profiles with the linear K–H eigenmode (as will be discussed in Section 6), but not necessarily linear dynamics of the underlying physics. The overestimation past the first few diameters is attributed to strong differences between the measured fluctuations and the K–H eigenmode alone. If an accurate projection of the experimental data were performed using a large number of eigenmodes, perfect agreement with the hot-wire velocity would be recovered again, as the eigenmodes are a complete set [27]. However, an accurate bi-orthogonal projection comprising a large number of eigenmodes is not possible here due to the limitations associated with the use of experimental data; as discussed in Section 5.1, the errors of the projection of experimental data for other eigenmode branches are of the same order of magnitude as the cofactor and prevent an accurate bi-orthogonal projection.

Further downstream the K–H eigenmode merges with the branch of mixing-layer modes, and projection results are not shown in Fig. 4.

5.3. Determination of amplitudes of PSE wavepacket models

Linear models are arbitrary with respect to a constant amplitude factor \( A_{\text{am}} \) that must be determined using additional data if the wavepacket models are required to serve as predictive tools. In previous works, the amplitudes were chosen ad hoc in order to obtain the best visual agreement with experimental measurements [9,12,13]. The bi-orthogonal projection of experimental data is an alternative, theoretically-founded method of determining this amplitude. Linear stability theory for parallel flows shows that the Kelvin–Helmholtz eigenmode is the only source of linear amplification of the fluctuations in subsonic jets. Even if there is a broad fluctuation spectra at the nozzle lip due to the turbulent mixing layer, the fluctuations associated with the K–H mechanism will grow and eventually dominate the PSE solution at a short distance downstream, at least while it remains unstable. Consequently, computing the projected amplitude of the experimental data on the K–H eigenmode at a single cross-section determines the PSE wavepacket amplitude completely [27,43].

Fig. 4. Power spectral density of the axial velocity fluctuation at the centerline: Hot-wire (dashed line); TR-PIV (crosses); projection on the K–H eigenmode (circles); projection on the K–H eigenmode ± variance (thin solid line).

\[
\begin{align*}
\text{(a) } St &= 0.1 \\
\text{(b) } St &= 0.2 \\
\text{(c) } St &= 0.3 \\
\text{(d) } St &= 0.4 \\
\text{(e) } St &= 0.5 \\
\text{(f) } St &= 0.6 \\
\text{(g) } St &= 0.7 \\
\text{(h) } St &= 0.8 \\
\text{(i) } St &= 0.9
\end{align*}
\]
Fig. 5. Comparison between the profiles of axial velocity component $\bar{u}_x$ at different cross-sections for Fourier mode $St = 0.5, m = 0$: Raw PIV data (dashed red); POD-filtered PIV data (solid red); projection on K–H eigenmode (dashed black); PSE solution (solid black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Not all cross-sections are adequate for the amplitude determination via adjoint projection. As seen in Fig. 4, only a region of few diameters from the nozzle delivers the correct amplitudes $a_n$ (Eq. (10)) for the K–H eigenmode and can be used to determine $A_{nm}$ (Eq. (4)). Fig. 10 shows the centerline velocity corresponding to the PSE wavepackets scaled by the projection of the TR-PIV data at a single cross-section. For all frequencies, the PSE computations are initiated at $x_0 = 0D$ while the amplitude determination is done at a different cross-section due to the technical impossibility of measuring at the nozzle exit cross-section. The axial location where the amplitude is determined for each frequency can be identified in the figure, as the PSE solution and the extracted K–H eigenmode must be nearly coincident there. A small deviation exists because the K–H eigenmode from LST and the PSE solution are not identical (comparisons are shown in Section 6).

6. Comparison of cross-stream velocity profiles

This section compares the velocity profiles measured by stereoscopic time-resolved PIV with those corresponding to the locally-parallel LST and PSE computations, discussed in the preceding sections. Again, TR-PIV data is Fourier transformed in the azimuthal direction and time, allowing comparisons for each $(St, m)$ pair.

In a previous work [13], proper orthogonal decomposition (POD) [48,49] was used in order to extract the velocity fluctuations most coherent on each cross-section from the PIV measurements. Whereas the unfiltered experimental velocity profiles presented some discrepancies with the computed PSE velocity fields, the leading POD mode, containing most of the fluctuating kinetic energy, was shown to be in good agreement with PSE up to the potential core closure. The reader is referred to Cavalieri et al. [13] for a detailed presentation of the procedure to obtain the POD modes from the experimental data.

In the following comparisons, only the first, most energetic POD mode for each $(St, m)$ is considered:

$$POD(\bar{u}) = \sqrt{\lambda^{(1)}}\xi^{(1)},$$

(11)

where $\lambda^{(1)}$ is the highest eigenvalue and $\xi^{(1)}$ the corresponding eigenfunction. This projection is used to extract the coherent part of the velocity field, and is referred to as the ‘POD mode’ throughout the remainder of this paper.

6.1. Axisymmetric mode at $St = 0.5$

Comparisons of the axial and radial velocity profiles for $St = 0.5$ and $m = 0$ at different axial sections are shown in Figs. 5 and 6, respectively. The POD and PSE solution agree closely in the first diameters from the nozzle lip, losing similarity gradually as we move downstream. In the first diameters, the K–H eigenmode from LST compares reasonably well to the PSE solution both in shape and amplitude. This agreement starts deteriorating when the K–H eigenmode predicted by the LST problem becomes neutral and then stable, as discussed in Section 5. For $St = 0.5, m = 0$ the K–H eigenmode becomes stable at $x \approx 3D$. Afterwards, the velocity profiles corresponding to the K–H eigenmode differ from the PSE solution. This discrepancy should be expected, as the PSE integration tends to follow the dominant (i.e. most unstable or less damped) eigenmode in the local LST eigenspectrum. When the K–H eigenmode becomes stable at the beginning of the Region II, discussed in Section 4, PSE no longer delivers an approximation of the discrete K–H eigenmode but a combination of different eigenmodes.
including part of the mixing-layer and core vorticity modes, that are nearly neutral or have a smaller decay rate than the K–H. Consequently, PSE computations agree better with the TR-PIV profiles than the K–H eigenmode alone once the physical mechanism responsible for the wavepacket amplification, the Kelvin–Helmholtz instability, is no longer active. As the contribution of many additional eigenmodes becomes relevant, the bi-orthogonal projection on the K–H eigenmode does not deliver meaningful predictions any more.

6.2. First helical mode at \( St = 0.5 \)

Comparisons of the axial development of the velocity profiles for the mode \( St = 0.5, m = 1 \) are done in Fig. 7. Again, reasonable agreement is found between the K–H eigenmode and the PSE solution for the first two diameters, that is gradually lost as the K–H eigenmode at this \( St \) and wavenumber becomes stable at \( x \approx 3D \). The agreement is also good with the first POD mode. Further downstream, PSE resembles the POD mode until the end of the potential core both in shape and amplitude.

6.3. Axisymmetric mode at \( St = 0.2 \)

Linear PSE wavepacket models have consistently been found to compare poorly with experiments and simulations of unforced turbulent jet flows at frequencies below \( St \approx 0.3 \) [12, 13, 31]. The axial amplification of the linear PSE solution clearly underestimates the measured evolution, as shown in Fig. 4. At least two possible explanations were proposed to this mismatch [12]. One possibility is that the separation of scales required by the PSE approach is not verified by the lower frequencies, as the characteristic wavelengths become comparable to the length of the potential core. The second possibility is that the neglected inter-modal nonlinear interactions, that seem to have very limited effect at higher frequencies, are important at the lower frequency range. The present results suggest that nonlinearity is the cause of the disagreement, as discussed next.

Fig. 4 shows that projection of the experimental fluctuations on the K–H eigenmode overestimates the amplitude of the hot-wire measurements by a constant factor for every cross-section, an error that can be attributed to aliasing in the TR-PIV measurements. However, the downstream amplification of the projected K–H eigenmode follows closely the one observed in the hot-wire measurements, up to \( x \approx 4.5D \). The agreement in the slopes is more evident for \( St = 0.1 \). Note, again, that the evolution of the amplitude of the projected K–H eigenmode is not governed by the growth rate predicted by LST, but only by the changes in the eigenfunction and in the experimental data. The amplification predicted by LST at \( x = 1.5D \) is shown by straight gray lines in Fig. 10 for comparison, and is similar to the amplification predicted by linear PSE.

Comparisons of the axial and radial velocity profiles for \( St = 0.2 \) and \( m = 0 \) at different axial sections are shown respectively in Figs. 8 and 9. The shape of the PSE and K–H eigenmode profiles agree for most of the cross-sections, while differences in the amplitude appear as we move downstream. Comparisons with the POD mode are not as good as for \( St = 0.5 \), but the main features of the profile, such as the linsle peak and the phase jump about \( r = 0.7D \) agree reasonably well. The similarities found in the velocity profiles predicted by linear models (LST and PSE) with the experimental data suggests that the mean flow divergence is not responsible for the underestimation of the spatial amplification, but that nonlinear interactions force the growth of the instability wave [50]. Otherwise, important differences would appear between the locally-parallel LST and PSE, as the latter is affected by the spread of the mean mixing layer.
Fig. 7. Comparison between the profiles of axial velocity component $\bar{u}_x$ at different cross-sections for Fourier mode $St = 0.5, m = 1$: Raw PIV data (dashed red); POD-filtered PIV data (solid red); projection on K–H eigenmode (dashed black); PSE solution (solid black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

On the other hand, some of the characteristics observed are shared with weakly nonlinear models usually employed in the study of laminar–turbulent transition. In the context of transitional flow, weakly nonlinear regimes are characterized by the amplification of subharmonic modes by nonlinear forcing due to another frequency, the evolution of which remains approximately linear. In these cases, the fluctuation profiles remain similar to those of LST in conditions under which the fluctuation amplitudes are finite and relevant nonlinear excitation of the subharmonics exist. The possibility of this kind of subharmonic excitation in turbulent jets was demonstrated by Suponitsky et al. [51].

7. On the relation between locally-parallel LST and the wavepacket evolution

The changes in the degree of agreement found in the comparison of the amplitudes and velocity profiles along the axial direction shown in the previous sections can be directly related to the variations of the LST eigenspectrum properties discussed in Section 4, that are in turn associated with changes in the mean flow. The ability of PSE to model the axial evolution of the large-scale wavepackets can also be explained in the perspective of LST results. While the notion of eigenmode does not hold strictly within the PSE framework, the PSE Ansatz and its iterative solution algorithm implies that the perturbations computed tend to follow the evolution dictated by the dominant parallel-flow eigenmodes. In this manner, the fluctuation field recovered by PSE can be described, in an approximate and qualitative manner, in terms of known physical mechanisms.

In Section 4 three spatial regions were identified, attending to the value of the parameter $\omega_0/D$ englobing the mixing-layer thickness and frequency. In this section, the division in spatial regions is revisited in view of the results presented above. Fig. 10 repeats for clarity some of the results in Fig. 4, also showing (vertical dashed lines) the approximate boundaries of the different regions for each Strouhal number considered.

Region I is related to the mean flow’s initial mixing layer and extends up to the transition region where $\omega_0/D = O(1)$. This region is characterized by a top-hat mean velocity profile, and LST predicts that the K–H eigenmode is unstable. Good agreement is found between the PSE and K–H velocity profiles in the first diameter from the nozzle, but they differ from the experimental ones. Incidentally, the amplitude of the projection on the K–H eigenmode agrees well with the experiment, and imposing this amplitude to the PSE solution results in reasonable comparisons with the experimental wavepackets downstream, as seen in Fig. 10. These observations suggest that another zone, Region 0, exists in the experiments, that extends approximately over the first diameter. This region corresponds to a transition on the fluctuations that evolve from oscillations internal to the nozzle towards the inflectional instability observed downstream in Region I. The portion of the fluctuations that are unrelated to the wavepacket decay while the wavepacket is amplified due to inflectional instability, gradually improving the agreement between the K–H eigenfunction, the PSE solution and the first POD mode.

Region II corresponds to a transition region in which the mixing layer becomes relatively thick, comparable to the radial extent of the potential core where $\omega_0/D = O(1)$. A single K–H eigenmode can still be identified in the LST eigenspectrum, but it is stable and eventually (as we move downstream towards thicker mixing layers or the frequency is increased) merges with a branch of eigenmodes whose fluctuations are localized in the relatively thick mixing layer. As the K–H eigenmode is no longer the dominant eigenmode, i.e. the mechanism responsible for the wavepacket amplification
Fig. 8. Comparison between the profiles of axial velocity component $\hat{u}_x$ at different cross-sections for Fourier mode $St = 0.2$, $m = 0$: Raw PIV data (dashed red); POD-filtered PIV data (solid red); projection on K–H eigenmode (dashed black); PSE solution (solid black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

ceases, the PSE solution is no longer governed by it and recovers a combination of other eigenmodes that decay slower. In this region, the wavepackets constitute the dominant velocity fluctuations in the experiment and POD is an efficient method for their elucidation. The first POD mode compares well with the PSE solution, but the K–H eigenfunction does not. This observation explains the differences between models based on the K–H eigenmode alone [6–8,47] and PSE computations past the amplitude peak (Regions II and III), while they are nearly coincident in Region I.

Region III corresponds to the developed turbulent jet, starting when $\delta$ becomes much bigger than $R$ ($\delta \sim 1$ and $R \ll 1$) and extending downstream. With the disappearance of the potential core, the branches of mixing-layer and core eigenmodes coalesce, and the PSE solution is again a mixture of different eigenmodes. The agreement between the linear PSE model and the first POD mode extracted from the measured profiles is lost in this region. Cavalieri et al. [13] pointed out that one possible explanation for the disagreement is the existence of important nonlinearities towards the end of the potential core. The present results do not help in clarifying this question, but related works considering subsonic unforced jets [12,21] showed good agreement between near-field pressure measurements and the same linear PSE solutions discussed here, even downstream of the potential core. These results suggest that the wavepackets persist in the region of developed turbulent jet and that the linear PSE models are indeed representative of the velocity fluctuations associated with them, but account for a relatively small amount of the kinetic energy making their identification in the velocity field difficult.

8. Conclusions

In a recent work [13], we investigated the existence and nature of wavepackets in the velocity field of unforced turbulent jets with subsonic exit velocity. The present work investigates further this question by performing a detailed study of the properties of the parallel-flow LST eigenspectrum and its evolution along the axial direction. Subsequently, time-resolved PIV measurements at cross sections are used in a LST-based bi-orthogonal projection to educe the fluctuations associated with the Kelvin–Helmholtz eigenmode. The other eigenmode families were not considered due to the limitations inherent to the use of experimental data. Comparisons between the fluctuation profiles corresponding to experimental data with the projected K–H eigenmode and the PSE solution, and their evolution along the axial direction, are explained in terms of changes in the LST eigenspectrum.

It is found that the amplification driven by inflectional instability of the mean flow dominates the first diameters of evolution of the wavepackets. Saturation occurs as a result of the mixing-layer spread and takes place when the parameter $\omega \delta/D$, governing the local linear instability, becomes of order unity. The Kelvin–Helmholtz mechanism is then no longer active, and the wavepacket evolution cannot be accurately modeled by a single discrete eigenmode. Wavepacket models based on the discrete Kelvin–Helmholtz eigenmode alone, as parallel-flow LST [47] or multiple-scale approaches [6,8] overestimate the decay rates of the wavepackets. On the other hand, PSE does not follow a particular eigenmode but tends to converge towards the least damped eigenmodes in the LST eigenspectrum, delivering solutions that compare well with experiments even in the decay region.

The projection of the measured fluctuations on the dominant instability eigenmode presented in this work enabled the study of two additional aspects of the PSE wavepacket models. Linear PSE computations have been shown to deliver good comparisons with wavepacket signatures at frequencies $St \geq 0.3$, while under-predicting the growth at lower frequencies. The agreement found
Fig. 9. Comparison between the profiles of radial velocity component $\bar{u}$ at different cross-sections for Fourier mode $St = 0.2$, $m = 0$: Raw PIV data (dashed red); POD-filtered PIV data (solid red); projection on K–H eigenmode (dashed black); PSE solution (solid black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 10. Power spectral density of the axial velocity fluctuation at the centerline: Hot-wire (dashed line); projection on the K–H eigenmode (circles); PSE solution (solid line). The vertical dashed lines correspond, respectively, to the axial location where K–H eigenmode becomes stable, where K–H eigenmode coalesces with the branch of shear-layer eigenmodes, and where the branches of shear-layer and core vorticity modes merge. The gray line shows the slope predicted by the local LST problem at $x = 1.5D$. 
between the amplification of centerline velocity in hot-wire measurements and the projected $K-H$ eigenmode over some diameters suggests that nonlinear excitation may be significant at these low frequencies, as suggested by Suponitsky et al. [51].

Finally, the bi-orthogonal projection of empirical data on the Kelvin–Helmholtz eigenmode is shown to be a valid method for the determination of the wavepacket amplitudes in linear PSE models. The convective nature of the inflectional instability that gives rise to the wavepackets implies that, once the amplitudes at a single near-nozzle cross-section are determined, the PSE models predict the axial evolution of the wave packet amplitudes without further knowledge of the downstream evolution of the fluctuations.

Acknowledgments

We thank Drs. Kristjan Gudmundsson, Anrab Samanta and Aniruddha Sinha for their contributions on the development of the PSE code and Carine Fourment-Cazenave, Patrick Braud and Dr. Joël Delville for their work during the experiments. The work of D. Rodríguez was supported by the European Union Marie Curie COFUND programme.

References